Elliptic Curves, Algebraic Geometry Approach in Gravity Theory and Uniformization of Multivariable Cubic Algebraic Equations

Bogdan G. Dimitrov *
Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research
6 Joliot-Curie str.
Dubna 141 980, Russia

Abstract

Based on the distinction between the covariant and contravariant metric tensor components in the framework of the affine geometry approach and the s.c. "gravitational theories with covariant and contravariant connection and metrics", it is shown that a wide variety of third, fourth, fifth, seventh, tenth- degree algebraic equations exists in gravity theory. This is important in view of finding new solutions of the Einstein's equations, if they are treated as algebraic ones. Since the obtained cubic algebraic equations are multivariable, the standard algebraic geometry approach for parametrization of two-dimensional cubic equations with the elliptic Weierstrass function cannot be applied. Nevertheless, for a previously considered cubic equation for reparametrization invariance of the gravitational Lagrangian and on the base of a newly introduced notion of "embedded sequence of cubic algebraic equations", it is demonstrated that in the multivariable case such a parametrization is also possible, but with complicated irrational and non-elliptic functions. After finding the solutions of a system of first - order nonlinear differential equations, these parametrization functions can be considered also as uniformization ones (depending only on the complex uniformization variable z) for the initial multivariable cubic equation.

1 INTRODUCTION

Inhomogeneous cosmological models have been intensively studied in the past in reference to colliding gravitational waves [1] or singularity structure and generalizations of the Bondi

*Electronic mail: bogdan@theor.jinr.ru

- Tolman and Eardley-Liang-Sachs metrics [2, 3]. In these models the inhomogeneous metric is assumed to be of the form [2]

$$ds^2 = dt^2 - e^{2\alpha(t,r,y,z)}dr^2 - e^{2\beta(t,r,y,z)}(dy^2 + dz^2) \tag{1.1}$$

(or with $r \rightarrow z$ and $z \rightarrow x$), which is called the Szafron-Szekeres metric [4-7]. In [7], after an integration of one of the components - G_1^0 of the Einstein's equations, a solution in terms of an elliptic function is obtained.

In different notations, but again in the framework of the Szafron-Szekerez approach the same integrated in [7] nonlinear differential equation

$$\left(\frac{\partial \Phi}{\partial t}\right)^2 = -K(z) + 2M(z)\Phi^{-1} + \frac{1}{3}\Lambda\Phi^2 \tag{1.2}$$

was obtained in the paper [8] of Kraniotis and Whitehouse. They make the useful observation that (1.2) in fact defines a (cubic) algebraic equation for an elliptic curve, which according to the standard algebraic geometry prescribtions (see [9] for an elementary, but comprehensive and contemporary introduction) can be parametrized with the elliptic Weierstrass function

$$\rho(z) = \frac{1}{z^2} + \sum_{\omega}\left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right] \tag{1.3}$$

and the summation is over the poles in the complex plane. Two important problems immediately arise, which so far have remained without an answer:

1. The parametrization procedure with the elliptic Weierstrass function in algebraic geometry is adjusted for cubic algebraic equations with number coefficients! Unfortunately, equation (1.2) is not of this type, since it has coefficient functions in front of the variable Φ, which depend on the complex variable z. In view of this, it makes no sense to define "Weierstrass invariants" as

$$g_2 = \frac{K^2(z)}{12} \quad ; \quad g_3 = \frac{1}{216}K^3(z) - \frac{1}{12}\Lambda M^2(z) \quad , \tag{1.4}$$

since the above functions have to be set up equal to the complex numbers g_2 and g_3 (the s. c. Eisenstein series)

$$g_2 = 60\sum_{\omega \subset \Gamma}\frac{1}{\omega^4} = \sum_{n,m}\frac{1}{(n+m\tau)^4} \quad , \tag{1.5}$$

$$g_3 = 140\sum_{\omega \subset \Gamma}\frac{1}{\omega^6} = \sum_{n,m}\frac{1}{(n+m\tau)^6} \tag{1.6}$$

and therefore additional equations have to be satisfied in order to ensure the parametrization with the Weierstrass function.

2. Is the Szekerez - Szafron metric the only case, when the parametrization with the Weierstrass function is possible? Closely related to this problem is the following one - is

only one of the components of the Einstein's equation parametrizable with $\rho(z)$ and its derivative?

This paper has the aim to present an adequate mathematical algorithm for finding solutions of the Einstein's equations in terms of elliptic functions. This approach is based on the clear distinction between covariant and contravariant metric tensor components within the s.c affine geometry approach, which will be clarified further in Section 2. Afterwords, a cubic algebraic equation in terms of the contravariant metric components will be obtained, which according to the general prescription and the algorithm in the previous paper [10] can be parametrized with the Weierstrass function and its derivative. Respectively, if the contravariant components are assumed to be known, then a cubic (or a quartic) algebraic equation with respect to the covariant components can be investigated and parametrized again with the Weierstrass function. *Thus it will turn out that the parametrization with the Weierstrass function will be possible not only in the Szafron-Szekeres case, but also in the general case due to the "cubic" algebraic structure of the gravitational Lagrangian.* This is an important point since valuable cosmological characteristics for observational cosmology such as the Hubble's constant $H(t) = \frac{\dot{R}(t)}{R(t)}$ and the deceleration parameter $q = -\frac{\ddot{R}(t)R(t)}{\dot{R}^2(t)}$ may be expressed in terms of the Jacobi's theta function and of the Weierstrass elliptic function respectively [8]. Unfortunately, in the paper [8] the Eisenstein series (1.5-1.6) have not been taken into account, due to which the obtained expression for the metric will be another one and will be modified.

Instead of searching out eliptic solutions of the Einstein's equations for each separate case of a given metric, as in nearly all of the mentioned papers, in the this paper another method will be proposed. First, a cubic algebraic equataion will be parametrized with respect to one of the contravariant components, following the approach in a previous paper [10]. Then, this parametrization will be extended to more than one variable in the *multivariable cubic algebraic equation (section 6).* This will be a substantial and new development, different from the standard algebraic geometry approach, in which *only two-dimensional cubic equations* are parametrized with the (elliptic) Weierstrass function and its derivative. Finally, the dependence of the generalized coordinates $X^i = X^i(x_1, x_2, x_3, \ldots.., x_n)$ on the complex variable z will be established from a derived system of first-order nonlinear differential equations (section 7). The generalized coordinates can be regarded as $n-$ dimensional hypersurfaces, defining a transition from an initially defined set of coordinates $x_1, x_2, x_3, \ldots.., x_n$ on a chosen manifold to another set of the generalized coordinates $X^1, X^2, \ldots.., X^n$. Since the covariant metric components g_{ij} also depend on these coordinates, this means that their dependence on the complex variable z will also be known. In other words, at the end of the applied approach, each initially given function $g_{ij}(t, \mathbf{x})$ of the time and space coordinates will be expressed also as $g_{ij}(z)$. The algebraic approach will be applied to the s .c. *cubic algebraic equation for reparametrization invariance of the gravitational Lagrangian*, but further it will be shown that not only the approach will be applicable in the general case of an arbitrary contravariant tensor, but also concrete solutions for the metric $g_{ij}(z)$ will be given in the case of specially chosen simple metrics.

The present paper continues and develops further the approach from a previous paper [10], where a definite choice of the contravariant metric tensor was made in the form of the factorized product $\widetilde{g}^{ij} = dX^i dX^j$. The differentials dX^i are assumed to lie in the tangent space T_X of the generalized coordinates. In Section 2 of the present paper some basic facts about the affine geometry approach and the s.c. *gravitational theory with covariant and contravariant metrics and connections (GTCCMC)* will be reminded, which has been described in the review article [13]. In its essence, the distinction between covariant and contravariant components is related to the *affine geometry approach* [15, 16], according to which the four-velocity tangent vector at each point of the observer's worldline is *not normalized* and *not equal to one*, i.e. $l_a l^a = l^2 \neq 1$. Similarly, for a second-rank tensor one would have $g_{\mu\nu} g^{\nu\alpha} = l^{\alpha}_{\mu} \neq \delta^{\alpha}_{\mu}$. In the next section 3 it will be demonstrated briefly how the cubic algebraic equation with respect to the differentials dX^i was derived in [10], but in fact the aim will be to show that depending on the choice of variables in the gravitational Lagrangian or in the Einstein's equations, a *wide variety of algebraic equations* (of third, fourth, fifth, seventh degree) in gravity theory may be treated, if a *distinction between the covariant metric tensor components and the contravariant ones is made.* This idea, originally set up by Schouten and Schmutzer, was further developed in the papers [13, 14]. In usual gravity theory, the contravariant components are at the same time inverse to the covariant ones , and thus the correspondence between "covectors" (in our terminology - these are the "vectors") and the "vectors" (i.e. the contravariant vectors") is being set up, since both these kinds of tensors satisfy the matrix equation $g_{ij} g^{jk} = \delta^k_i$. However, within the framework of affine geometry, such a correspondence is not necessarily to be established (see again [15-18]) and both tensors have to be treated as *different mathematical objects,* defined on one and the same manifold.

The physical idea, which will be exploited in this paper will be: *can such a gravitational theory with a more general contravariant tensor have the same gravitational Lagrangian as in the known gravitational theory with contravariant metric tensor components, which are at the same time the inverse ones to the covariant one?* On the base of such an "equivalence" the s. c. cubic algebraic equation for reparametrization invariance of the gravitational Lagrangian was obtained in [10]. The derivation was based also on the construction of another connection $\widetilde{\Gamma}^s_{kl} \equiv \frac{1}{2} dX^i dX^s (g_{ik,l} + g_{il,k} - g_{kl,i})$. It can be proved that the connection $\widetilde{\Gamma}^s_{kl}$ has two very useful properties: 1. It may have an affine transformation law under a broad variety of coordinate transformations, which can be found after solving a system of nonlinear differential equations. 2. $\widetilde{\Gamma}^s_{kl}$ is an equiaffine connection, which is a typical notion, introduced in classical affine geometry [15, 16] and meaning that there exists a volume element, which is preserved under a parallel displacement of a basic $n-$dimensional vector $e \equiv e_{i_1 i_2 \ldots i_n}$. Equivalently defined, $\widetilde{\Gamma}^s_{kl}$ is an equiaffine connection [15, 16] if it can be represented in the form $\widetilde{\Gamma}^s_{ks} = \partial_k lge$, where e is a scalar quantity. This notion turns out to be very convenient and important, since for such types of connections we can use the known formulae for the Ricci tensor, but with the connection $\widetilde{\Gamma}^s_{kl}$ instead of the usual Christoffell one Γ^s_{kl}. Moreover, the Ricci tensor $\widetilde{R}_{ij}$ will again be a symmetric one, i.e. $\widetilde{R}_{ij} = \widetilde{R}_{ji} = \partial_k \widetilde{\Gamma}^k_{ij} - \partial_i \widetilde{\Gamma}^k_{kj} + \widetilde{\Gamma}^k_{kl} \widetilde{\Gamma}^l_{ij} - \widetilde{\Gamma}^m_{ki} \widetilde{\Gamma}^k_{jm}$.

2 AFFINE GEOMETRY APPROACH AND GRAVITATIONAL THEORIES WITH COVARIANT AND CONTRAVARIANT METRICS AND CONNECTIONS

This section has the purpose to review some of the basic aspects of *GTCCMC*, which would further allow the application of algebraic geometry and theory of algebraic equations in gravity theory.

It is known in gravity theory that the metric tensor g_{ij} determines the space - time geometry, which means that the Christoffell connection

$$\Gamma_{ik}^{l} \equiv \frac{1}{2} g^{ls}(g_{ks,i} + g_{is,k} - g_{ik,s}) \tag{2.1}$$

and the Ricci tensor

$$R_{ik} = \frac{\partial \Gamma_{ik}^{l}}{\partial x^{l}} - \frac{\partial \Gamma_{il}^{l}}{\partial x^{k}} + \Gamma_{ik}^{l} \Gamma_{lm}^{m} - \Gamma_{il}^{m} \Gamma_{km}^{l} \tag{2.2}$$

can be calculated.

It is useful to remember also [23] the s. c. Christoffell connection of the first kind:

$$\Gamma_{i;kl} \equiv g_{im} \Gamma_{kl}^{m} = \frac{1}{2}(g_{ik,l} + g_{il,k} - g_{kl,i}) \quad , \tag{2.3}$$

obtained from the expression for the zero covariant derivative $0 = \nabla_{l} g_{ik} = g_{ik,l} - g_{m(i}\Gamma_{k)l}^{m}$. By contraction of (2.3) with another contravariant tensor field $\widetilde{g}^{is}$, one might as well define *another connection*:

$$\widetilde{\Gamma}_{kl}^{s} \equiv \widetilde{g}^{is} \Gamma_{i;kl} = \widetilde{g}^{is} g_{im} \Gamma_{kl}^{m} = \frac{1}{2} \widetilde{g}^{is}(g_{ik,l} + g_{il,k} - g_{kl,i}) \quad , \tag{2.4}$$

not consistent with the initial metric g_{ij}. Clearly the connection (2.4) is defined under the assumption that the contravariant metric tensor components $\widetilde{g}^{is}$ are not to be considered to be the inverse ones to the covariant components g_{ij} and therefore $\widetilde{g}^{is} g_{im} \equiv f_{m}^{s}(\mathbf{x})$.

In fact, the definition $\widetilde{g}^{is} g_{im} \equiv f_{m}^{s}$ turns out to be inherent to gravitational physics. For example, in the projective formalism one decomposes the standardly defined metric tensor (with $g_{ij} g^{jk} = \delta_{i}^{k}$) as

$$g_{ij} = p_{ij} + h_{ij} \quad , \tag{2.5}$$

together with the additional assumption that the two subspaces, on which the projective tensor p_{ij} and the tensor h_{ij} are defined, are orthogonal. This means that

$$p_{ij} h^{jk} = 0 \quad . \tag{2.6}$$

As a consequence

$$p_{ij} p^{jk} = \delta_{i}^{k} - h_{ij} h^{jk} \neq \delta_{i}^{k} \quad , \tag{2.7}$$

meaning that the contravariant projective metric components p^{jk} (in the orthogonal subspace to the tensor h_{ij}) are no longer inverse to the covariant ones p_{ij}.

An example of gravitational theories with more than one connection are the so called *theories with affine connections and metrics* [13], in which there is one connection $\Gamma^{\gamma}_{\alpha\beta}$ for the case of a parallel transport of covariant basic vectors $\nabla_{e_\beta} e_\alpha = \Gamma^{\gamma}_{\alpha\beta}\ e_\gamma$ and a *separate* connection $P^{\gamma}_{\alpha\beta}$ for the contravariant basic vector e^γ, the defining equation for which is $\nabla_{e_\beta} e^\alpha = P^{\alpha}_{\gamma\beta}\ e^\gamma$. In these theories, the contravariant vector and tensor fields are assumed to be *not the inverse ones* to the covariant vector and tensor fields. This implies that

$$e_\alpha e^\beta \equiv f^\beta_\alpha(x) \neq \delta^\beta_\alpha \tag{2.8}$$

and consequently, a distinction is made between *covariant* and *contravariant* metric tensors (and vectors too). Clearly, in the above given case (2.7) of projective gravity, this theory should be considered as a GTCCMC. In the same spirit, since the well - known Arnowitt - Deser - Misner (ADM) (3+1) decomposition of spacetime [43, 44] is built upon the projective transformation (2.5), it might be thought that it should also be considered as such a theory. But in fact, the ADM (3+1) formalism definitely is not an example for this *due to the special identification of the vector field's components [43, 44] with certain components of the projective tensor, as a consequence of which all the (spacelike defined) contravariant projective tensor components $p^{\alpha\beta}$ $(\alpha, \beta, \gamma = 0, 1, 2, 3\ ;\ \ i, j = 1, 2, 3)$ turn out to be the inverse ones to the covariant projective components $p_{\alpha\gamma}$.* In the case of the ADM (3+1) decomposition, such an identification is indeed possible and justified, since the gravitational field posseses coordinate invariance, allowing to disentangle the dynamical degrees of freedom from the gauge ones. But in the case when the tensors h_{ij} are related with some moving matter (with a prescribed motion) and an observer, "attached" to this matter "measures" all the gravitational phenomena in his reference system by means of the projective metric p_{ij}, this will be no longer possible. Then the relation (2.7) will hold, and the resulting theory will be a *GTCCMC*. Naturally, if the tensor h_{ij} in (2.5) and (2.7) is taken in the form $h_{ij} = \frac{1}{e} u_i u^j$ and if the vector field u (tangent at each point of the trajectory of the moving matter) is assumed to be non-normalized (i.e. $e(x) = u_i u^i \neq 1$), then one would have to work *not within* the standard relativistic hydrodynamics theory (where $p_{ij} = g_{ij} - u_i u_j$ and $p_{ij} p^{jk} = \delta^k_i - u_i u^k$), but within the formalism of GTCCMC (where $p_{ij} p^{jk} = f^k_i = \delta^k_i - \frac{1}{e} u_i u^k \neq \delta^k_i$). One may wonder why this should be so, since the last two formulaes for $p_{ij} p^{jk}$ for the both cases look very much alike, with the exception of the "normalization" function $\frac{1}{e}$ in the second formulae. But it shall become clear that in the *first case* the right-hand side has a *tensor transformation property*, while in the *second case* due to the function $\frac{1}{e}$ there would be no such property. And this shall turn out to be crucial.

In order to understand this also from another point of view, let us perform a covariant differentiation of both sides of the relation (2.8). Then one can obtain that the two connections are related in the following way [13]

$$f^i_{j,k} = \Gamma^l_{jk}\ f^i_l + P^i_{lk} f^l_j \qquad ; \qquad (f^i_{j,k} = \partial_k f^i_j) \qquad . \tag{2.13}$$

Note also the following important moment - $f_\alpha^\beta(x)$ are considered to be the components of a *function.* Otherwise, if they are considered to be a (mixed) tensor quantity, the covariant differentiation of the mixed tensor $f_\alpha^\beta(x)$ in the right-hand side of $e_\alpha e^\beta \equiv f_\alpha^\beta(x)$ would give exactly the same quantity as the one in the left-hand side for *every choice* of the two connections Γ_{jk}^l and P_{lk}^i, including also for the standard case (Einsteinian gravity) $P_{lk}^i = -\Gamma_{jk}^l$. This would mean that from a mathematical point of view there would be no justification for the introduction of the second covariant connection P_{lk}^i . However, since $f_\alpha^\beta(x)$ are related with the description of some moving matter in the Universe, a tensor transformation law should not be prescribed to them. So they should remain components of a function and consequently, the introduction of the second connection P_{lk}^i is inevitable.

In order to understand further why and in *what cases the distinction between covariant and contravariant metric components will lead to an inevitable introduction of two different connections* Γ_{ij}^k *and* P_{ij}^k, let us prove the following statement:

Proposition 1 *If* $e_1, e_2, ..., e_n$ *is a basis of covariant vector fields and* f_i^α *are the components of a given function or a constant, then a basis of contravariant basic fields* $\widetilde{e}^{\alpha_1}, \widetilde{e}^{\alpha_2}, ..., \widetilde{e}^{\alpha_n}$ *can be found so that for each* i *and* α_j *one has* $e_i \widetilde{e}^{\alpha_j} = f_i^\alpha$.

This proposition is in fact is a generalization of the well-known theorem from differential geometry that if a basis of (covariant) vector fields is given, then a dual basis of (contravariant) vector fields can be found, so that the contravariant vector fields are the *inverse ones to the covariant* ones, i.e. $e_i \widetilde{e}^{\alpha_j} = \delta_i^\alpha$.

The proof is very simple, but essentially based on the relation (2.13). If the covariant basic vector fields are given, then the contravariant connection components Γ_{ij}^k will be known too. Since $f_{j,k}^i$ are derivatives of a function, one may take the expression (2.13) $f_{j,k}^i = \Gamma_{jk}^l \, f_l^i + P_{lk}^i f_j^l$, which for the moment shall be treated as a system of $n.[\frac{n(n+1)}{2}]$ *linear algebraic equations with respect to the (unknown) connection components* P_{lk}^i. A solution of this system can be found for the connection components P_{lk}^i. Then the condition for the parallel transport of the contravariant basic vector fields $\nabla_{e_\beta} \widetilde{e}^\alpha = P_{\gamma\beta}^\alpha \, \widetilde{e}^\gamma$ can be written as $\partial_\beta \widetilde{e}^\alpha = P_{\gamma\beta}^\alpha \, \widetilde{e}^\gamma$ and considered as a system of n *ordinary differential equations with respect to the components* $\widetilde{e}^\alpha$. From the solution of this system, $\widetilde{e}^\alpha$ can be found. So the new basis $\widetilde{e}^{\alpha_1}, \widetilde{e}^{\alpha_2}, ..., \widetilde{e}^{\alpha_n}$ will be *uniquely determined, up to the integration constants,* contained in the solution of the system of differential equations.

After proving this proposition, the difference between standard relativistic hydrodynamics and *"modified" relativistic hydrodynamics with a variable length* can be easily understood. In the first case, the right-hand side in $p_{ij} p^{jk} = \delta_i^k - u_i u^k = f_i^k \neq \delta_i^k$ transforms as a tensor, which is ensured also by normalization property $u_i u^i = 1$.Therefore (2.13) and the proposition will not hold, so the contravariant basic vector fields are determined in the standard way $e_i e^j = \delta_i^j$ and more importantly, they *cannot be determined* in another way, in spite of the fact that $f_i^k \neq \delta_i^k$.

In the second case, the situation is just the opposite - the right-hand side of $p_{ij} p^{jk} = \delta_i^k - \frac{1}{e} u_i u^k = f_i^k \neq \delta_i^k$ transforms not as a tensor because of the "normalization" factor

$\frac{1}{e}$, the proposition holds and thus the basic vector fields are determined as $e_i\widetilde{e}^j = f_i^j$. *Therefore, the treatment of relativistic hydrodynamics with "variable length" should be within the GTCCMC.*

In the present case, the introduced new connection (2.4) *should not be identified* with the connection $P^{\gamma}_{\alpha\beta}$, since the connection $\widetilde{\Gamma}^s_{kl} \equiv \widetilde{g}^{is}\Gamma_{i;kl}$ is introduced by means of modifying the contravariant tensor and not on the base of any separately defined parallel transport for the contravariant basic vectors. Moreover, the connection $\widetilde{\Gamma}^s_{kl}$ turns out to be a linear combination of the Christoffell connection components $\Gamma^{\gamma}_{\alpha\beta}$, and the relation between them is not of the type (2.13). In such a way, there will not be a contradiction with the case when the *two connections $\Gamma^{\gamma}_{\alpha\beta}$ and $\widetilde{\Gamma}^s_{kl}$ are not defined as separate ones*, since later on, in deriving the cubic algebraic equation in the general case and for the case $\widetilde{g}^{jk} = dX^j dX^k$ also, it would be supposed that $\widetilde{g}^{is}$ is a tensor. This would mean (from $\widetilde{g}^{is}g_{im} \equiv f^s_m(\mathbf{x})$) that $f^s_m(\mathbf{x})$ will also be a (mixed) tensor quantity, and therefore the covariant differentiation of $e_{\alpha}e^{\beta} \equiv f^{\beta}_{\alpha}(x)$ will not produce any new relation.

3 BASIC ALGEBRAIC EQUATIONS IN GRAVITY THEORY. TENSOR LENGTH SCALE

If one writes down the Ricci tensor in terms of the newly defined contravariant tensor $\widetilde{g}^{ij} \equiv dX^i dX^j$, then the following fourth - degree algebraic equation can be obtained

$$R_{ik} = dX^l \left[g_{is,l}\frac{\partial(dX^s)}{\partial x^k} - \frac{1}{2}p g_{ik,l} + \frac{1}{2}g_{il,s}\frac{\partial(dX^s)}{\partial x^k} \right] +$$

$$+\frac{1}{2}dX^l dX^m dX^r dX^s \left[g_{m[k,t}g_{l]r,i} + g_{i[l,t}g_{mr,k]} + 2g_{t[k,i}g_{mr,l]} \right] \qquad , \tag{3.1}$$

where p is the scalar quantity

$$p \equiv div(dX) \equiv \frac{\partial(dX^l)}{\partial x^l}, \tag{3.2}$$

which "measures" the divergency of the vector field dX. The algebraic variety of the equation consists of the differentials dX^i and their derivatives $\frac{\partial(dX^s)}{\partial x^k}$.

In the same spirit, one can investigate the problem whether the gravitational Lagrangian in terms of the new contravariant tensor can be equal to the standard representation of the gravitational Lagrangian. This standard *(first) representation* of the gravitational Lagrangian is based on the standard Christoffell connection Γ^k_{ij} (given by formulae (2.1)), the Ricci tensor R_{ik} (formulae (2.2)) and the *other contravariant tensor* $\widetilde{g}^{ij} = dX^i dX^j$ [10]

$$L_1 = -\sqrt{-g}\widetilde{g}^{ik}R_{ik} = -\sqrt{-g}dX^i dX^k R_{ik} \quad . \tag{3.3}$$

In the *second representation,* the Christoffell connection $\widetilde{\Gamma}^k_{ij}$ and the Ricci tensor $\widetilde{R}_{ik}$ are "tilda" quantities, meaning that the "tilda" Christoffell connection is determined by

formulae (2.4) with the new contravariant tensor $\widetilde{g}^{ij} = dX^i dX^j$ and the "tilda" Ricci tensor $\widetilde{R}_{ik}$ - by formulae (2.2), but with the "tilda" connection $\widetilde{\Gamma}_{ij}^{k}$ instead of the usual Christoffell connection Γ_{ij}^{k}. Thus the expression for the *second representation* of the gravitational Lagrangian acquires the form

$$L_2 = -\sqrt{-g}\widetilde{g}^{il}\widetilde{R}_{il} = -\sqrt{-g}dX^i dX^l \{p\Gamma_{il}^{r} g_{kr} dX^k - \Gamma_{ik}^{r} g_{lr} d^2X^k - \Gamma_{l(i}^{r} g_{k)r} d^2X^k\} \ . \quad (3.4)$$

The condition for the *equivalence of the two representations* $L_1 = L_2$ gives a cubic algebraic equation with respect to the algebraic variety of the first differential dX^i and the second ones d^2X^i [10]

$$dX^i dX^l \left(p\Gamma_{il}^{r} g_{kr} dX^k - \Gamma_{ik}^{r} g_{lr} d^2X^k - \Gamma_{l(i}^{r} g_{k)r} d^2X^k\right) - dX^i dX^l R_{il} = 0 \qquad . \quad (3.5)$$

In [22] also another cubic algebraic equation has been obtained, but after the application of a variational approach.

Following the approach in [10], the Einstein's equations in vacuum for the general case were derived under the assumption that the contravariant metric tensor components are the "tilda" ones:

$$0 = \widetilde{R}_{ij} - \frac{1}{2} g_{ij}\widetilde{R} = \widetilde{R}_{ij} - \frac{1}{2} g_{ij} dX^m dX^n \widetilde{R}_{mn} =$$

$$= -\frac{1}{2} p g_{ij} \Gamma_{mn}^{r} g_{kr} dX^k dX^m dX^n + \frac{1}{2} g_{ij} (\Gamma_{km}^{r} g_{nr} + \Gamma_{n(m}^{r} g_{k)r}) d^2X^k dX^m dX^n +$$

$$+ p\Gamma_{ij}^{r} g_{kr} dX^k - (\Gamma_{ik}^{r} g_{jr} + \Gamma_{j(i}^{r} g_{k)r}) d^2X^k \qquad . \quad (3.6)$$

This equation represents again a system of cubic equations. In addition, if the differentials dX^i and d^2X^i are known, but not the covariant tensor g_{ij}, the same equation can be considered also as a cubic algebraic equation with respect to the algebraic variety of the metric tensor components g_{ij} and their first derivatives $g_{ij,k}$.

It might be thought that the definite choice of the contravariant tensor in the form of the factorized product $\widetilde{g}^{ij} = dX^i dX^j$ is a serious restriction, in view of the fact that the second derivatives of the covariant tensor components $g_{ij,kl}$ are not present in the equation. This is indeed so, because the algebraic structure of the equation is simpler to deal with in comparison with the general case, and so it is easier to implement the algorithm for parametrization, developed in [10]. But there is one argument in favour of this choice (although the case for an arbitrary contravariant tensor is no doubt more important) - since the metric can be expressed as $ds^2 = l(x) = g_{ij} dX^i dX^j$ (consequently $dX^i dX^j = l(x) g^{ij}$), the obtained cubic algebraic equations (3.5) and (3.6) can be considered in regard also to the length function $l(x)$. Since for Einsteinian gravity $g_{ij} g^{jk} = \delta_i^k$ (i.e. $g^{jk} = \widetilde{g}^{jk} = dX^j dX^k$), then for this case the length function is "postulated" to be $l = 1$. But the length function can also be obtained as a solution of the cubic equation, and thus in more general theories of gravity solutions with $l \neq 1$ may exit. In fact, for a general contravariant tensor $\widetilde{g}^{ij} \neq dX^i dX^j$, one would have $\widetilde{g}^{ij} = l_k^i g^{kj}$, where l_k^i will be proposed

to be called a *"tensor length scale"*, and the previously defined length function $l(x)$ is a partial case of the tensor length scale for $l_j^i = l\delta_j^i$. The *physical meaning* of the notion of tensor length scale is simple - in the different directions (i.e. for different i and j) the length scale is different. In particular, some motivation for this comes from Witten's paper [45], where in discussing some aspects of weakly coupled heterotic string theory (when there is just one string couplings) and the obtained too large bound on the Newton's constant it was remarked that *"the problem might be ameliorated by considering an anisotropic Calabi - Yau with a scale $\sqrt{\alpha'}$ in d directions and $\frac{1}{M_{GUT}}$ in $(6-d)$ directions"*. Thus it may be proposed to realize this if one takes

$$l_i^k = g_{ij}dX^j dX^k = l_1\delta_i^k \quad \text{for} \quad i, j, k = 1, ..., d \qquad , \tag{3.7}$$

$$l_a^b = g_{ac}dX^c dX^b = l_2\delta_a^b \quad \text{for} \quad a, b, c = d+1, ..., 6 \qquad . \tag{3.8}$$

Note also the justification for the name *"tensor length scale"* - if l_k^i is a tensor quantity, so will be the "modified" contravariant tensor $\widetilde{g}^{ij} = l_k^i g^{kj}$, and consequently in accord with section 2 there will be no need for the introduction of a new covariant connection P_{ij}^k. And this is indeed the case, because the relation between the two connections Γ_{ij}^k and $\widetilde{\Gamma}_{ij}^k$ is given by formulae (2.4) $\widetilde{\Gamma}_{kl}^s := \widetilde{g}^{is}g_{im}\Gamma_{kl}^m$. In other words, these two connections are not considered to be "separately introduced" and so they do not depend on one another by means of the equality (2.13).

The purpose of the present paper further will be: *how can one extend the proposed in [10] approach for the "modified" contravariant metric components (as $\widetilde{g}^{ij} = dX^i dX^j$) to the case of a generally defined contravariant tensor $\widetilde{g}^{ij} \neq dX^i dX^j$?*

4 INTERSECTING ALGEBRAIC VARIETIES AND STANDARD (EINSTEINIAN) GRAVITY THEORY

A more general theory with the definition of the contravariant tensor as $\widetilde{g}^{ij} \equiv dX^i dX^j$ should contain in itself the standard gravitational theory with $g_{ij}g^{jk} = \delta_i^k$. From a mathematical point of view, this should be performed by considering the intersection [19, 20, 21] of the cubic algebraic equations (3.6) with the system of n^2 quadratic algebraic equations for the algebraic variety of the n variables

$$g_{ij}dX^j dX^k = \delta_i^k \qquad . \tag{4.1}$$

In its general form $g_{ij}\widetilde{g}^{jk} = \delta_i^k$ with an arbitrary contravariant tensor $\widetilde{g}^{jk}$, this system can also be considered together with the Einstein's "algebraic" system of equations, which in the next section shall be derived for a *generally defined contravariant tensor.* From an algebro - geometric point of view, this is the problem about the intersection of the Einstein's algebraic equations with the system of n^2 (linear) hypersurfaces for the $\left[\begin{pmatrix} n \\ 2 \end{pmatrix} + n\right]$

contravariant variables, if the covariant tensor components are given. Since the derived Einstein's algebraic equations are again cubic ones with respect to the contravariant metric components, this is an analogue to the well - known problem in algebraic geometry about the intersection of a (two-dimensional) cubic curve with a straight line. However, in the present case the straight line and the cubic curve are *multi - dimensional ones*, which is a substantial difference from the standard case in algebraic geometry.

The standardly known solutions of the Einstein's equations can be obtained as an intersection variety of the Einstein's algebraic equations with the system $g_{ij}\widetilde{g}^{jk} = \delta_i^k$. However, the strict mathematical proof that such an intersection will give the known solutions is still lacking.

5 ALGEBRAIC EQUATIONS FOR A GENERAL CONTRAVARIANT METRIC TENSOR

Let us write down the algebraic equations for all admissable parametrizations of the gravitational Lagrangian for the generally defined contravariant tensor $\widetilde{g}^{ij}$, following *the same prescription* as in section 3, where the equality of the two representations of the gravitational Lagrangian has been supposed:

$$\widetilde{g}^{i[k}\widetilde{g}^{l]s}_{,l}\Gamma^r_{ik}g_{rs} + \widetilde{g}^{i[k}\widetilde{g}^{l]s}\left(\Gamma^r_{ik}g_{rs}\right)_{,l} +$$
$$+\widetilde{g}^{ik}\widetilde{g}^{ls}\widetilde{g}^{mr}g_{pr}g_{qs}\left(\Gamma^q_{ik}\Gamma^p_{lm} - \Gamma^p_{il}\Gamma^q_{km}\right) - R = 0 \quad . \tag{5.1}$$

This equation is again a *cubic algebraic equation* with respect to the algebraic variety of the variables $\widetilde{g}^{ij}$ and $\widetilde{g}^{ij}_{,k}$, and the number of variables in the present case is much greater than in the previous case for the contravariant tensor $\widetilde{g}^{ij} \equiv dX^i dX^j$. At the same time, this equation is a *fourth - degree* algebraic equation with respect to the covariant metric tensor g_{ij} and its first and second partial derivatives. With respect to the algebraic variety of all the variables $\widetilde{g}^{ij}$, $\widetilde{g}^{ij}_{,k}$, g_{ij}, $g_{ij,k}$, $g_{ij,kl}$, the above algebraic equation is of *seventh order* and with coefficient functions, due to the presence of the terms with the affine connection Γ^q_{ik} and its derivatives, which contain the contravariant tensor g^{ij} and $g^{ij}_{,k}$.

Similarly, the Einstein's equations can be written as a system of *third - degree algebraic equations* with respect to the (generally chosen) contravariant variables and their derivatives

$$0 = \widetilde{R}_{ij} - \frac{1}{2}g_{ij}\widetilde{R} =$$
$$= \widetilde{g}^{lr}(\Gamma_{r;i[j}),_{l]} + \widetilde{g}^{lr}_{,[l}\Gamma_{r;ij]} + \widetilde{g}^{lr}\widetilde{g}^{ms}(\Gamma_{r;ij}\Gamma_{s;lm} - \Gamma_{s;il}\Gamma_{r;km}) -$$
$$-\frac{1}{2}g_{ij}\widetilde{g}^{m[k}\widetilde{g}^{l]s}_{,l}\Gamma^r_{mk}g_{rs} - \frac{1}{2}g_{ij}\widetilde{g}^{m[k}\widetilde{g}^{l]s}\left(\Gamma^r_{mk}g_{rs}\right)_{,l} -$$
$$-\frac{1}{2}g_{ij}\widetilde{g}^{nk}\widetilde{g}^{ls}\widetilde{g}^{mr}g_{pr}g_{qs}\left(\Gamma^q_{nk}\Gamma^p_{lm} - \Gamma^p_{nl}\Gamma^q_{km}\right) \quad . \tag{5.2}$$

Interestingly, the same system of equations can be considered as a system of *fifth - degree* equations with respect to the covariant variables (which is the difference from the previous

case). The mathematical treatment of fifth - degree equations is known since the time of Felix Klein's famous monograph [24], published in 1884. A way for resolution of such equations on the base of earlier developed approaches by means of reducing the fifth - degree equations to the so called modular equation has been presented in the more recent monograph of Prasolov and Solov'yev [9]. Some other methods for solution of third-, fifth- and higher- order algebraic equations have been given in [25, 26]. A complete description of elliptic, theta and modular functions has been given in the old monographs [27, 28]. Also, solutions of $n-$ th degree algebraic equations in theta - constants [29] and in special functions [30] are interesting in view of the not yet proven hypothesis in the paper by Kraniotis and Whitehouse [8] that *"all nonlinear solutions of general relativity are expresed in terms of theta - functions, associated with Riemann - surfaces"*. Some other monographs, related to elliptic functions and elliptic curves are [31-42].

Two other important problems can be pointed out:

1. One can find solutions of the system of Einstein's equations not as solutions of a system of nonlinear differential equations, but as elements of an algebraic variety, satisfying the Einstein's algebraic equations. The important new moment is that this gives an opportunity to find solutions of the Einstein's equations both for the components of the covariant metric tensor g_{ij} and for the contravariant ones $\widetilde{g}^{jk}$. This means that solutions may exist for which $g_{ij}\widetilde{g}^{jk} \neq \delta_i^k$. In other words, a classification of the solutions of the Einstein's equations can be performed in an entirely new and nontrivial manner - under a given contravariant tensor, the covariant tensor and its derivatives have to be found from the algebraic equation, or under a given covariant tensor, the contravariant tensor and its derivatives can be found.

2. The condition for the zero - covariant derivative of the covariant metric tensor $\nabla_k g_{ij} = 0$ and of the contravariant metric tensor $\nabla_k \widetilde{g}^{ij} = 0$ can be written in the form of the following cubic algebraic equations with respect to the variables g_{ij}, $g_{ij,k}$ and $\widetilde{g}^{ls}$ **:**

$$\nabla_k g_{ij} \equiv g_{ij,k} - \widetilde{\Gamma}^l_{k(i} g_{j)l} = g_{ij,k} - \widetilde{g}^{ls} \Gamma_{s;k(i} g_{j)l} = 0 \tag{5.3}$$

and

$$0 = \nabla_k \widetilde{g}^{ij} = \widetilde{g}^{ij}_{,k} + \widetilde{g}^{r(i} \widetilde{g}^{j)s} \Gamma_{r;sk} \quad . \tag{5.4}$$

The first equation (5.3) is linear with respect to $\widetilde{g}^{ls}$ and quadratic with respect to g_{ij}, $g_{ij,k}$, while the second equation (5.3) is linear with respect to g_{ij}, $g_{ij,k}$ and quadratic with respect to $\widetilde{g}^{ls}$.

6 EMBEDDED SEQUENCE OF ALGEBRAIC

EQUATIONS AND FINDING THE SOLUTIONS OF THE CUBIC ALGEBRAIC EQUATION

The purpose of the present subsection will be to describe the method for finding the solution (i. e . the algebraic variety of the differentials dX^i) of the cubic algebraic equation (3.5) (in the limit $d^2X^k = 0$). The method has been proposed first in [10] but here it will be developed further and applied with respect to a *sequence of algebraic equations* with algebraic varieties, which are embedded into the initial one. This means that if at first the algorithm is applied with respect to the three-dimensional cubic algebraic equation (3.5) and a solution for dX^3 (depending on the Weierstrass function and its derivative) is found, then the same algorithm will be applied with respect to the two-dimensional cubic algebraic equation with variables dX^1 and dX^2, and finally to the one-dimensional cubic algebraic equation of the variable dX^1 only.

The basic and very simple idea about parametrization of a cubic algebraic equation with the Weierstrass function [9, 11,12] can be presented as follows: Let us define the lattice $\Lambda = \{m\omega_1 + n\omega_2 \mid m, n \in Z; \omega_1, \omega_2 \in C, Im\frac{\omega_1}{\omega_2} > 0\}$ and the mapping $f : C/\Lambda \rightarrow CP^2$, which maps the factorized (along the points of the lattice Λ) part of the points on the complex plane into the two dimensional complex projective space CP^2. This means that each point z on the complex plane is mapped onto the point $(x, y) = (\rho(z), \rho^{'}(z))$, where x and y belong to the affine curve

$$y^2 = 4x^3 - g_2 x - g_3 \qquad . \tag{6.1}$$

In other words, the functions $x = \rho(z)$ and $y = \rho^{'}(z)$, where $\rho(z)$ denotes the *Weierstrass elliptic function* (1.3), are *uniformization functions* for the cubic curve. It can be proved [9] that the only cubic algebraic curve with number coefficients, which is parametrized by the uniformization functions $x = \rho(z)$ and $y = \rho^{'}(z)$ is the affine curve (6.1).

In the case of the cubic equation of reparametrization invariance (3.5), the aim will be again to bring the equation to the form (6.1) and afterwards to make equal each of the coefficient functions to the (numerical) coefficients in (6.1).

In order to provide a more clear description of the developed method, let us divide it into several steps.

Step 1. The initial cubic algebraic equation (3.5) is presented as a cubic equation with respect to the variable dX^3 only

$$A_3(dX^3)^3 + B_3(dX^3)^2 + C_3(dX^3) + G^{(2)}(dX^2, dX^1, g_{ij}, \Gamma_{ij}^k, R_{ik}) \equiv 0 \qquad , \tag{6.2}$$

where naturally the coefficient functions A_3, B_3 , C_3 and $G^{(2)}$ depend on the variables dX^1 and dX^2 of the algebraic subvariety and on the metric tensor g_{ij}, the Christoffel connection Γ_{ij}^k and the Ricci tensor R_{ij}:

$$A_3 \equiv 2p\Gamma_{33}^r g_{3r} \quad ; \qquad B_3 \equiv 6p\Gamma_{\alpha 3}^r g_{3r} dX^{\alpha} - R_{33} \quad , \tag{6.3}$$

$$C_3 \equiv -2R_{\alpha 3}dX^{\alpha} + 2p(\Gamma^r_{\alpha\beta}g_{3r} + 2\Gamma^r_{3\beta}g_{\alpha r})dX^{\alpha}dX^{\beta} \quad . \tag{6.4}$$

The Greek indices α, β take values $\alpha, \beta = 1, 2$ while the indice r takes values $r = 1, 2, 3$.

Step 2. A linear-fractional transformation

$$dX^3 = \frac{a_3(z)\widetilde{dX}^3 + b_3(z)}{c_3(z)\widetilde{dX}^3 + d_3(z)} \tag{6.5}$$

is performed with the purpose of setting up to zero the coefficient functions in front of the highest (third) degree of $\widetilde{dX}^3$. This will be achieved if $G^{(2)}(dX^2, dX^1, g_{ij}, \Gamma^k_{ij}, R_{ik}) = -\frac{a_3 Q}{c_3^3}$, where

$$Q \equiv A_3 a_3^2 + C_3 c_3^2 + B_3 a_3 c_3 + 2c_3 d_3 C_3 \quad . \tag{6.6}$$

This gives a cubic algebraic equation with respect to the two-dimensional algebraic variety of the variables dX^1 and dX^2:

$$p\Gamma^r_{\gamma(\alpha}g_{\beta)r}dX^{\gamma}dX^{\alpha}dX^{\beta} + K^{(1)}_{\alpha\beta}dX^{\alpha}dX^{\beta} + K^{(2)}_{\alpha}dX^{\alpha} + 2p\left(\frac{a_3}{c_3}\right)^3\Gamma^r_{33}g_{3r} = 0 \tag{6.7}$$

and $K^{(1)}_{\alpha\beta}$ and $K^{(2)}_{\alpha}$ are the corresponding quantities [10]

$$K^{(1)}_{\alpha\beta} \equiv -R_{\alpha\beta} + 2p\frac{a_3}{c_3}(1 + 2\frac{d_3}{c_3})(2\Gamma^r_{\alpha\beta}g_{3r} + \Gamma^r_{3\alpha}g_{\beta r}) \tag{6.8}$$

and

$$K^{(2)}_{\alpha} \equiv 2\frac{a_3}{c_3}\left[3p\frac{a_3}{c_3}\Gamma^r_{\alpha 3}g_{3r} - (1 + 2\frac{d_3}{c_3})R_{\alpha 3}\right] \quad . \tag{6.9}$$

Note that since the linear fractional transformation (with another coefficient functions) will again be applied with respect to another cubic equations, everywhere in (6.5 - 6.8) the coefficient functions $a_3(z)$, $b_3(z)$, $c_3(z)$ and $d_3(z)$ bear the indice "3", to distinguish them from the indices in the other linear-fractional tranformations, which are to be applied. In terms of the new variable $n_3 = \widetilde{dX}^3$ the original cubic equation (3.5) acquires the form [10]

$$\widetilde{n}^2 = \overline{P}_1(\widetilde{n})\ m^3 + \overline{P}_2(\widetilde{n})\ m^2 + \overline{P}_3(\widetilde{n})\ m + \overline{P}_4(\widetilde{n})\ , \tag{6.10}$$

where $\overline{P}_1(\widetilde{n})$, $\overline{P}_2(\widetilde{n})$, $\overline{P}_3(\widetilde{n})$ and $\overline{P}_4(\widetilde{n})$ are complicated functions of the ratios $\frac{c_3}{d_3}$, $\frac{b_3}{d_3}$ and A_3, B_3, C_3 (but not of the ratio $\frac{a_3}{d_3}$, which is very important). The variable m denotes the ratio $\frac{a_3}{c_3}$ and the variable $\widetilde{n}$ is related to the variable n through the expresssion

$$\widetilde{n} = \sqrt{k_3}\sqrt{C_3}\left[n + L^{(3)}_1\frac{B_3}{C_3} + L^{(3)}_2\right] \quad , \tag{6.11}$$

where

$$k_3 \equiv \frac{b_3}{d_3}\frac{c_3}{d_3}(\frac{c_3}{d_3} + 2) \quad , \tag{6.12}$$

$$L_1^{(3)} \equiv \frac{1}{2}\frac{\frac{b_3}{d_3}}{\frac{c_3}{d_3}+2} \quad ; \quad L_2^{(3)} \equiv \frac{1}{\frac{c_3}{d_3}+2} \quad . \tag{6.13}$$

The subscript "3" in $L_1^{(3)}$ and $L_2^{(3)}$ means that the corresponding ratios in the right-hand side also have the same subscript. Setting up the coefficient functions $\overline{P}_1(\widetilde{n})$, $\overline{P}_2(\widetilde{n})$, $\overline{P}_3(\widetilde{n})$ equal to the number coefficients $4, 0, -g_2, -g_3$ respectively, one can now parametrize the resulting equation

$$\widetilde{n}^2 = 4m^3 - g_2 m - g_3 \tag{6.14}$$

according to the standard prescription

$$\widetilde{n} = \rho^{'}(z) = \frac{d\rho}{dz} \qquad ; \qquad \frac{a_3}{c_3} \equiv \ m = \rho(z) \quad . \tag{6.15}$$

Taking this into account, representing the linear-fractional transformation (6.5) as (dividing by c_3)

$$dX^3 = \frac{\frac{a_3}{c_3}\widetilde{dX}^3 + \frac{b_3}{c_3}}{\widetilde{dX}^3 + \frac{d_3}{c_3}} \tag{6.16}$$

and combining expressions (6.11) for $\widetilde{n}$ and (6.16), one can obtain the final formulae for dX^3 as a solution of the cubic algebraic equation

$$dX^3 = \frac{\frac{b_3}{c_3} + \frac{\rho(z)\rho^{'}(z)}{\sqrt{k_3}\sqrt{C_3}} - L_1^{(3)}\frac{B_3}{C_3}\rho(z) - L_2^{(3)}\rho(z)}{\frac{d_3}{c_3} + \frac{\rho^{'}(z)}{\sqrt{k_3}\sqrt{C_3}} - L_1^{(3)}\frac{B_3}{C_3} - L_2^{(3)}} \quad . \tag{6.17}$$

In order to be more precise, it should be mentioned that the identification of the functions $\overline{P}_1(\widetilde{n})$, $\overline{P}_2(\widetilde{n})$, $\overline{P}_3(\widetilde{n})$ with the number coefficients gives some additional equations [10], which in principle have to be taken into account in the solution for dX^3. This has been investigated to a certain extent in [10], and will be continued to be investigated. Here in this paper the main objective will be to show the dependence of the solutions on the Weierstrass function and its derivative. Since only the ratios $\frac{b}{d}$ and $\frac{c}{d}$ enter these additional relations, and not $\frac{a}{c}$ (which is related to the Weierstrass function), they do not affect the solution with respect to $\rho(z)$ and $\rho^{'}(z)$.

Since B_3 and C_3 depend on dX^1 and dX^2, the solution (6.17) for dX^3 shall be called *the embedding solution* for dX^1 and dX^2.

Step 3. Let us now consider the two-dimensional cubic equation (6.6). Following the same approach and finding the "reduced" cubic algebraic equation for dX^1 only, it shall be proved that the solution for dX^2 is the embedding solution for dX^1.

For the purpose, let us again write down eq. (6.6) in the form (6.2), singling out the variable dX^2:

$$A_2(dX^2)^3 + B_2(dX^2)^2 + C_2(dX^2) + G^{(1)}(dX^1, g_{ij}, \Gamma_{ij}^k, R_{ik}) \equiv 0 \quad , \tag{6.18}$$

where the coefficient functions A_2, B_2, C_2 and $G^{(1)}$ are the following:

$$A_2 \equiv 2p\Gamma_{22}^r g_{2r} \quad ; \qquad B_2 \equiv K_{22}^{(1)} + 2p[2\Gamma_{12}^r g_{2r} + \Gamma_{22}^r g_{1r}]dX^1 \quad , \tag{6.19}$$

$$C_2 \equiv 2p[\Gamma^r_{11}g_{2r} + 2\Gamma^r_{12}g_{1r})(dX^1)^2 + (K^{(1)}_{12} + K^{(1)}_{21})dX^1 + K^{(2)}_2 \quad , \tag{6.20}$$

$$G^1 \equiv 2p\Gamma^r_{11}g_{1r}(dX^1)^3 + K^{(1)}_{11}(dX^1)^2 + K^{(2)}_1 dX^1 + 2p\rho^3(z)\Gamma^r_{33}g_{3r} \quad . \tag{6.21}$$

Note that the starting equation (6.7) has the same structure of the first terms, if one makes the formal substitution $-R_{\alpha\beta} \rightarrow K^{(1)}_{\alpha\beta}$ in the second terms, but eq. (6.7) has two more additional terms $K^{(2)}_1 dX^1 + 2p\rho^3(z)\Gamma^r_{33}g_{3r}$. Therefore, one might guess how the coefficient functions will look like just by taking into account the above substitution and the contributions from the additional terms. Revealing the general structure of the coefficient functions might be particularly useful in higher dimensions, when one would have a "chain" of cubic algebraic equations. Concretely for the three-dimensional case, investigated here, C_2 in (6.20) can be obtained from C_3 in (6.4), observing that there will be an additional contribution from the term $K^{(2)}_\alpha dX^\alpha$ for $\alpha = 2$. Also, in writing down the coefficient functions in (6.2) it has been accounted that as a result of the previous parametrization $\frac{a_3}{c_3} = \rho(z)$.

Since eq. (6.18) is of the same kind as eq. (6.2), for which we already wrote down the solution, the expression for dX^2 will be of the same kind as in formulae (6.17), but with the corresponding functions A_2, B_2, C_2 instead of A_3, B_3, C_3. Taking into account (6.19 - 6.20), the solution for dX^2 can be written as follows:

$$dX^2 = \frac{\frac{1}{\sqrt{k_2}}\rho(z)\rho^{'}(z)\sqrt{C_2} + h_1(dX^1)^2 + h_2(dX^1) + h_3}{\frac{1}{\sqrt{k_2}}\rho^{'}(z)\sqrt{C_2} + l_1(dX^1)^2 + l_2(dX^1) + l_3} \quad , \tag{6.22}$$

where $h_1, h_2, h_3, l_1, l_2, l_3$ are expressions, depending on $\frac{b_2}{d_2}, \frac{d_2}{c_2}, \Gamma^r_{\alpha\beta}$ $(r = 1, 2, 3\ ;\ \alpha, \beta = 1, 2)$, $g_{\alpha\beta}$, $K^{(1)}_{12}$, $K^{(1)}_{21}$ and on the Weierstrass function.

The representation of the solution for dX^2 in the form (6.22) shows that it is *an "embedding" solution* of dX^1 in the sense that it depends on this function.Correspondingly, the solution (6.17) for dX^3 is an *"embedding"* one for the variables dX^1 and dX^2.

Step 4. It remains now to investigate the one-dimensional cubic algebraic equation

$$A_1(dX^1)^3 + B_1(dX^1)^2 + C_1(dX^1) + G^{(0)}(g_{ij}, \Gamma^k_{ij}, R_{ik}) \equiv 0 \quad , \tag{6.23}$$

obtained from the two-dimensional cubic algebraic equation (6.18) after applying the linear-fractional transformation

$$dX^2 = \frac{a_2(z)\widetilde{dX}^2 + b_2(z)}{c_2(z)\widetilde{dX}^2 + d_2(z)} \tag{6.24}$$

and setting up to zero the coefficient function before the highest (third) degree of $(dX^2)^3$. Taking into account that as a result of the previous parametrization $\frac{a_2}{c_2} = \rho(z)$, the coefficient functions $A_1, B_1, C_1$and D_1 are given in a form, not depending on dX^2 and dX^3:

$$A_1 \equiv 2p\Gamma^r_{11}g_{1r} \quad , \tag{6.25}$$

$$B_1 \equiv F_3\rho(z) + K_{11}^{(1)} = 2p(1 + 2\frac{d_2}{c_2})[2\Gamma_{12}^r g_{1r} + \Gamma_{11}^r g_{2r}]\rho(z) + K_{11}^{(1)} \quad , \tag{6.26}$$

$$C_1 \equiv F_1\rho^2(z) + F_2\rho(z) + K_1^{(2)} = 2p[2\Gamma_{12}^r g_{2r} + \Gamma_{22}^r g_{1r}]\rho^2(z) + \\ +(1 + 2\frac{d_2}{c_2})(K_{12}^{(1)} + K_{21}^{(1)})\rho(z) + K_1^{(2)} \quad , \tag{6.27}$$

$$G^0 \equiv 2p[\Gamma_{22}^r g_{2r} + \Gamma_{33}^r g_{3r}]\rho^3(z) + K_{22}^{(1)}\rho^2(z) \quad . \tag{6.28}$$

The solution for dX^1 can again be written in the form (6.17), but with $\frac{b_1}{c_1}$, $\frac{d_1}{c_1}$, $L_1^{(1)}$, $L_2^{(1)}$, $k_1$and B_1, C_1 instead of these expressions with the indice "3".

Taking into account formulaes (6.25 - 6.28) for A_1, B_1 and C_1, the final expression for dX^1 can be written as

$$dX^1 = \frac{\frac{1}{\sqrt{k_1}}\rho(z)\rho^{'}(z)\sqrt{F_1\rho^2 + F_2\rho(z) + K_1^{(2)}} + f_1\rho^3 + f_2\rho^2 + f_3\rho + f_4}{\frac{1}{\sqrt{k_1}}\rho^{'}(z)\sqrt{F_1\rho^2(z) + F_2\rho(z) + K_1^{(2)}} + \widetilde{g}_1\rho^2(z) + \widetilde{g}_2\rho(z) + \widetilde{g}_3} \quad , \tag{6.29}$$

where $F_1, F_2, f_1, f_2, f_3, f_4, \widetilde{g}_1, \widetilde{g}_2$ and $\widetilde{g}_3$ are functions, depending on $g_{\alpha\beta}$, $\Gamma_{\alpha\beta}^r$ $(\alpha, \beta = 1, 2)$ and on the ratios $\frac{b_1}{c_1}$, $\frac{b_1}{d_1}$, $\frac{b_2}{d_2}$, $\frac{d_1}{c_1}$, $\frac{d_2}{c_2}$.

It is also straightforward to prove that expressions (6.22) for dX^2 and (6. 29) for dX^1do not represent elliptic functions. If one assumes that dX^1 is an elliptic function, then from the standard theory of elliptic functions [9, 11] it will follow that dX^1 can be represented as

$$dX^1 = K_1(\rho) + \rho^{'}(z)K_2(\rho) \quad , \tag{6.30}$$

where $K_1(\rho)$ and $K_2(\rho)$ depend on the Weierstrass function only. But this representation will contradict with the expression (6.29) for dX^1- consequently the initial assumption has to be rejected. Similarly, it can be proved that dX^2 is not an elliptic function. The details of this simple proof will be left for the interested reader.

7 COMPLEX COORDINATE DEPENDENCE OF THE METRIC TENSOR COMPONENTS FROM THE UNIFORMIZATION OF A CUBIC ALGEBRAIC SURFACE

In this section it will be shown that the solutions (6.17), (6. 22) and (6. 29) of the cubic algebraic equation (3.5) enable us to express not only the contravariant metric tensor components through the Weierstrass function and its derivatives, but the covariant components as well.

Let us write down for convenience the system of equations (6.17), (6. 22) and (6. 29) for dX^1, dX^2 and dX^3 as $(l = 1, 2, 3)$

$$dX^l(X^1, X^2, X^3) = F_l(g_{ij}(\mathbf{X}), \Gamma_{ij}^k(\mathbf{X}), \rho(z), \rho^{'}(z)) = F_l(\mathbf{X}, z) \quad , \tag{7.1}$$

where the appearence of the complex coordinate z is a natural consequence of the uniformization procedure, applied with respect to each one of the cubic equations from the "embedded" sequence of equations.

Yet how the appearence of the additional complex coordinate z on the right-hand side of (7.1) can be reconciled with the dependence of the differentials on the left-hand side only on the generalized coordinates (X^1, X^2, X^3) (and on the initial coordinates x^1, x^2, x^3 because of the mapping $X^i = X^i(x^1, x^2, x^3)$)? The only reasonable assumption will be that *the initial coordinates depend also on the complex coordinate*, i.e.

$$X^l \equiv X^l(x^1(z), x^2(z), x^3(z)) = X^l(\mathbf{x},\, z) \qquad . \tag{7.2}$$

Taking into account the important initial assumptions $(l = 1, 2, 3)$

$$d^2X^l = 0 = dF_l(\mathbf{X}(z), z) = \frac{dF_l}{dz}dz \quad , \tag{7.3}$$

one easily gets the system of three inhomogeneous linear algebraic equations with respect to the functions $\frac{\partial X^1}{\partial z}$, $\frac{\partial X^2}{\partial z}$ and $\frac{\partial X^3}{\partial z}$ $(l = 1, 2, 3)$:

$$\frac{\partial F_l}{\partial X^1}\frac{\partial X^1}{\partial z} + \frac{\partial F_l}{\partial X^2}\frac{\partial X^2}{\partial z} + \frac{\partial F_l}{\partial X^3}\frac{\partial X^3}{\partial z} + \frac{\partial F_l}{\partial z} = 0 \quad , \tag{7.4}$$

The solution of this algebraic system $(i, k, l = 1, 2, 3)$

$$\frac{\partial X^l}{\partial z} = G_l\left(\frac{\partial F_i}{\partial X^k}\right) = G_l\left(X^1, X^2, X^3, z\right) \tag{7.5}$$

represents a system of *three first - order nonlinear differential equations.* A solution of this system can always be found in the form

$$X^1 = X^1(z) \quad ; \quad X^2 = X^2(z) \quad ; \quad X^3 = X^3(z) \qquad . \tag{7.6}$$

and therefore, the metric tensor components will also depend on the complex coordinate z, i.e. $g_{ij} = g_{ij}(\mathbf{X}(z))$. Note that since the functions $\frac{\partial F_i}{\partial X^k}$ in the right-hand side of (7.5) depend on the Weierstrass function and its derivatives, it might seem natural to write that the solution of the above system of nonlinear differential equations g_{ij} will also depend on the Weierstrass function and its derivatives

$$g_{ij} = g_{ij}(X^1(\rho(z), \rho^{'}(z), X^2(\rho(z), \rho^{'}(z), X^3(\rho(z), \rho^{'}(z)) = g_{ij}(z) \quad . \tag{7.7}$$

Note however that for the moment we do not have a theorem that the solution of the system (7.5) will also contain the Weierstrass function**.** But in spite of this, the dependence on the complex coordinate z will be retained.

8 DISCUSSION

This paper continues the investigation of cubic algebraic equations in gravity theory, initiated in a previous paper [10].

Unlike in [10], where the treatment of cubic algebraic equations has been restricted only to the choice of the contravariant tensor $\widetilde{g}^{ij} = dX^i dX^j$, in the present paper it was demonstrated that under a more general choice of $\widetilde{g}^{ij}$, there is a wide variety of algebraic equations of various order, among which an important role play the cubic equations. Their derivation is based on two important initial assumptions:

1. The covariant and contravariant metric components are treated independently, which is a natural approach within the framework of *affine geometry* [15 - 18].

2. Under the above assumption, the gravitational Lagrangian (or Ricci tensor) should remain the same as in the standard gravitational theory with inverse contravariant metric tensor components.

The proposed approach allows to treat the Einstein's equations as algebraic equations, and thus to search for separate classes of solutions for the covariant and contravariant metric tensor components. It can be supposed also that the existence of such separate classes of solutions might have some interesting and unexplored until now physical consequences. It has been shown also that the "transition" to the standard Einsteinian theory of gravity can be performed by investigating the intersection with the corresponding algebraic equations.

The most important result in this paper is given in Section 6 and is related to the possibility to find the parametrization functions for a multicomponent cubic algebraic surface, again by consequent application of the linear-fractional transformation. The parametrization functions in this particular case represent complicated irrational expressions of the Weierstrass function and its derivative, unlike in the standard two - dimensional case, where they are the Weierstrass function itself and its first derivative. The advantage of applying the *linear- fractional transformations* (6.5) and (6.24) is that by adjusting their coefficient functions (so that the highest - third degree in the transformation equation will vanish), the following sequence of plane cubic algebraic equations is fulfilled (the analogue of eq.(65) in [10]):

$$P_1^{(3)}(n_{(3)})m_{(3)}^3 + P_2^{(3)}(n_{(3)})m_{(3)}^2 + P_3^{(3)}(n_{(3)})m_{(3)} + P_{(4)}^{(3)} = 0 \qquad , \tag{8.1}$$

$$P_1^{(2)}(n_{(2)})m_{(2)}^3 + P_2^{(2)}(n_{(2)})m_{(2)}^2 + P_3^{(2)}(n_{(2)})m_{(2)} + P_{(4)}^{(2)} = 0 \qquad , \tag{8.2}$$

$$P_1^{(1)}(n_{(1)})m_{(1)}^3 + P_2^{(1)}(n_{(1)})m_{(1)}^2 + P_3^{(1)}(n_{(1)})m_{(1)} + P_{(4)}^{(1)} = 0 \qquad , \tag{8.3}$$

where $m_{(3)}$, $m_{(2)}$, $m_{(1)}$ denote the ratios $\frac{a_3}{c_3}$, $\frac{a_2}{c_2}$, $\frac{a_1}{c_1}$ in the corresponding linear - fractional transformations and $n_{(3)}$, $n_{(2)}$, $n_{(1)}$ are the "new" variables $\widetilde{dX}^3$, $\widetilde{dX}^2$, $\widetilde{dX}^1$. The *sequence of plane cubic algebraic equations* (8.1 - 8.3) should be understood as follows: the first one (8.1) holds if the second one (8.2) is fulfilled; the second one (8.2) holds if the third one

(8.3) is fulfilled. Of course, in the case of n variables (i. e. n component cubic algebraic equation) the generalization is straightforward. Further, since each one of the above plane cubic curves can be transformed to the algebraic equation ($i = 1, 2, 3$)

$$\widetilde{n}^2_{(i)} = \overline{P}^{(i)}_1(\widetilde{n}_{(i)})m^3_{(i)} + \overline{P}^{(i)}_2(\widetilde{n}_{(i)})m^2_{(i)} + \overline{P}^{(i)}_3(\widetilde{n}_{(i)})m_{(i)} + \overline{P}^{(i)}_4(\widetilde{n}_{(i)}) \tag{8.4}$$

and subsequently to its parametrizable form, one obtains the solutions of the initial multicomponent cubic algebraic equation.

Finally, it has been shown that from the expressions (7.1) a system of first - order nonlinear differential equations can be obtained, for which always a solution $X^1 = X^1(z)$, $X^1 = X^1(z)$, $X^1 = X^1(z)$ exists. Thus the dependence on the generalized coordinates X^1**,** X^2**,** X^3 in the uniformization functions (7.1) dissappears and only the dependence on the complex coordinate z remains, as it should be for uniformization functions.

Acknowledgments

The author is grateful to Dr. I. B. Pestov (BLTP, JINR, Russia), Dr. D. M. Mladenov (Theor. Phys.Departm., Fac. of Physics, Sofia Univ., Bulgaria), and especially to Prof. V. V. Nesterenko (BLTP, JINR, Russia), Dr. O. Santillan (IAFE, Buenos Aires, Argentina) and Prof. Sawa Manoff (INRNE, BAS, Bulgaria) for valuable comments, discussions and critical remarks.

This paper is written in memory of Prof. S. S. Manoff (1943 - 27.05.2005) - a specialist in classical gravitational theory.

The author is grateful also to Dr. G. V. Kraniotis (Max Planck Inst., Munich, Germany) for sending me his published paper (ref. [8]).

References

[1] P. Szekerez 1972 *J. Math. Phys.* **13** (3) 286 - 294

[2] P. Szekerez 1975 *Commun. Math. Phys.* **41** 55 - 64

[3] D. Eardley, E. Liang, and R. Sachs 1972 *J. Math. Phys.* **13** (1) 99 - 107

[4] D. A. Szafron 1977 *J. Math. Phys.* **18** (8) 1673 - 1677

[5] D. A. Szafron, and J. Wainwright 1977 *J. Math. Phys.* **18** (8) 1668 - 1672

[6] S. W. Goode, and J. Wainwright 1982 *Phys. Rev.* **D26** (12) 3315 - 3326

[7] J. D. Barrow, and J. Stein - Schabez 1984 *Phys. Lett.* **103A** (6, 7) 315 -317

[8] G. V. Kraniotis, and S. B. Whitehouse 2002 *Class. Quant. Grav.* **19** 5073 - 5100 (*Preprint* gr-qc/0105022)

[9] V. V. Prasolov, and Y. P. Solov'yev 1997 *Elliptic Functions and Elliptic Integrals (AMS Translations of Mathematical Monographs 170)* (R.I.: Providence) [Russian original: V. V. Prasolov, and Y. P. Solov'yev 1997 *Elliptic Functions and Algebraic Equations* (Moscow: Factorial Publishing House)

[10] B. G. Dimitrov 2003 *J. Math. Phys.* **44 (6)** 2542 - 2578 (*Preprint* hep-th/0107231)

[11] A. Hurwitz, and R. Courant 1964 *Allgemeine Funktionentheorie und Elliptische Funktionen* (Berlin-Heidelberg: Springer-Verlag)

[12] A. Hurwitz, and R. Courant 1929 *Funktionentheorie* (Berlin: Verlag von J. Springer)

[13] S. Manoff 1999 *Part. Nucl.* **30** 517 - 549 [Rus. Edit. 1999 Fiz. Elem. Chast. Atomn.Yadra. **30** (5) 1211 - 1269] (*Preprint* gr-qc/0006024)

[14] S. Manoff 2004 *Phys. Part. Nucl.* **35** 633 - 674 [Russian Edit. 2004 *Fiz. Elem. Chast. Atom. Yadra* **35** 1185 - 1258] (*Preprint* gr-qc/0309050)

[15] A. P. Norden 1950 *Spaces of Affine Connection* (Moscow: Nauka Publ.House)

[16] P. A. Shirokov, and A. P. Shirokov 1959 *Affine Differential Geometry* (Moscow: Fizmatgiz)

[17] J. A. Wolf 1972 *Spaces of Constant Curvature* (Berkeley: University of California)

[18] W. Slebodzinski 1998 *Exterior Forms and Their Applications* (University of Beijing: College Press)

[19] W. V. D. Hodge, and D. Pedoe 1952 *Methods of Algebraic Geometry* (Cambridge:Cambridge University Press)

[20] R. J. Walker 1950 *Algebraic Curves* (New Jersey: Princeton University Press)

[21] V. V. Prasolov, and O. V. Schwarzman 1999 *The Alphabet of Riemann Surfaces* (Moscow: Fazis Publ. House)

[22] B. G. Dimitrov 2001, in *Perspectives of Complex Analysis, Differential Geometry and Mathematical Physics Proceedings of the 5th International Workshop on Complex Structures and Vector Fields, St. Konstantin, Bulgaria, 3 - 9 September 2000, eds. S. Dimiev, K. Sekigawa* (Singapore: World Scientific) p. 171 - 179 (*Preprint* gr-qc/0107089)

[23] L. Landau, and E. Lifschiz 1988 *Theoretical Physics, vol. II. Field Theory* (Moscow: Nauka)

[24] F. Klein 1884 *Vorlesungen Uber das Ikosaeder und Auflosung der Gleichungen vom Funften Grade* (Leipzig)

[25] L. K. Lahtin 1893 *Algebraic Equations, Solvable in Hypergeometric Functions* (Moscow: University Press) (in Russian)

[26] L. K. Lahtin 1897 *Differential Resolvents of Higher Order Algebraic Equations* (Moscow: University Press) (in Russian)

[27] H. Weber 1896 *Lehrbuch der Algebra. Band 2. Gruppen. Lineare Gruppen. Anwendungen der Gruppen Theorie. AlgebraischeZahlen* (Braunschweig)

[28] H. Weber 1908 *Lehrbuch der Algebra. Band 3. Elliptische Funktionen und Algebraische Zahlen* (Braunschweig)

[29] D. Mumford 1893 1894 *Tata Lectures on Theta vol. 1,2* (Boston-Basel-Stuttgart: Birkhauser)

[30] G. Belardinelli 1960 *Fonctions Hypergeometriques De Plusieurs Variables et Resolution Analytique Des Equations Algebriques Generales* (Paris: Gauthier - Villars)

[31] S. Stoilov 1954 *Theory of Functions of Complex Variables* (Editura Academiei Romans)

[32] N. I. Akhiezer 1979 *Elements of the Elliptic Functions Theory* (Moscow:Nauka Publish. House)

[33] F. Klein, und R. Fricke 1890 *Vorlesungen Uber Die Theorie Der Elliptischen Modulfunctionen Band I und II* (Leipzig: Druck und Verlag von B.G. Teubner)

[34] R. Fricke, und F. Klein 1926 *Vorlesungen Uber Die Theorie Der Automorphen Functionen Band I und II* (Leipzig- Berlin: Verlag und Druck von B. G. Teubner)

[35] J. Silverman, and J. Tate 1992 *Rational Points on Elliptic Curves* (New York: Springer - Verlag)

[36] D. Husemoller 1987 *Elliptic Curves (Graduate Texts in Mathematics 111)* (New York: Springer-Verlag)

[37] A. W. Knapp 1992 *Elliptic Curves (Mathematical Notes 40)* (Princeton N.J.: Princeton University Press)

[38] J. Silverman 1986 *The Arithmetic of Elliptic Curves* (New York: Springer-Verlag)

[39] J. Silverman 1994 *Advanced Topics in the Arithmetic of Elliptic Curves* (New York: Springer-Verlag)

[40] T. M. Apostol 1976 *Modular Functions and Dirichlet Series in Number Theory* (New York: Springer-Verlag)

[41] A. Robert 1973 *Elliptic Curves (Lecture Notes in Mathematics 326)* (New York: Springer-Verlag)

[42] M. Waldschmidt, P. Moussa, J.-M. Luck, and C. Itzykson (eds.) 1992 *From Number Theory to Physics* (Berlin Heidelberg: Springer-Verlag)

[43] R. Arnowitt, S. Deser, and C. Misner 1962 The Dynamics of General Relativity, in "*Gravitation: An Introduction to Current Research*", ed. by L. Witten (New York - London: John Wiley & Sons Inc.) (*Preprint* gr-qc/0405109*)*

[44] R. Arnowitt, S. Deser, and C. Misner 1959 Phys. Rev. **116** 1322

[45] E.Witten 1996 *Nucl. Phys.* **B 471** 135